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The Skyrmion strikes back: baryons and a new large N_c limit

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Abstract: In the large N_c limit of QCD, baryons can be modeled as solitons, for instance, as Skyrmions. This modeling has been justified by Witten's demonstration that all properties of baryons and mesons scale with $N_c^{-1/2}$ in the same way as the analogous meson-based soliton model scales with a generic meson-meson coupling constant g. An alternative large N_c limit (the orientifold large N_c limit) has recently been proposed in which quarks transform in the two-index antisymmetric representation of $SU(N_c)$. By carrying out the analog of Witten's analysis for the new orientifold large N_c limit, we show that baryons and solitons can also be identified in the orientifold large N_c limit. However, in the orientifold large N_c limit, the interaction amplitudes and matrix elements scale with N_c^{-1} in the same way as soliton models scale with the generic meson coupling constant g rather than as $N_c^{-1/2}$ as in the traditional large N_c limit.

Keywords: QCD, 1/N Expansion, Solitons Monopoles and Instantons.

Contents

1. Introduction

In 1973 't Hooft proposed a large N_c limit for QCD [1] that has proved to be a powerful tool in studying QCD and other strongly coupled gauge theories. 't Hooft's idea was to generalize the gauge group of QCD from $SU(3)$ to $SU(N_c)$, and take $N_c \to \infty$ while keeping $g^2 N_c$ and the number of flavors N_f fixed. In this limit quark loops are suppressed, and non-planar diagrams are suppressed by a factor of N_c^{-2} for each handle. This greatly reduces the number of diagrams one must consider and allows one to make many qualitative predictions. For instance, quark-loop suppression implies the OZI rule, and baryons can be treated as solitons in the large N_c limit [2−4]. While this helps explain an important qualitative feature of hadronic physics, it does pose a phenomenological difficulty in relating the large N_c limit to the physical world of $N_c = 3$. To wit, there are the important cases in which the OZI rule is badly violated, and they are not explained in the large N_c limit. These cases include the situations in which the $U(1)_A$ anomaly plays a critical role, such as in the $\eta' - \pi$ mass difference.

A new large N_c limit for QCD that was proposed by Armoni, Shifman, and Veneziano [6−9] has received considerable recent attention. This limit, which they have dubbed the 'orientifold large N_c limit', starts from the observation that at $N_c = 3$ a quark can be described in two equivalent ways. It can be described as a Dirac spinor field transforming according to the fundamental representation of color $SU(3)$ or, equivalently, as a Dirac spinor field transforming according to the two-index anti-symmetric representation of color $SU(3)$. One can take a large N_c limit starting from either one of these two possibilities. Starting from the fundamental representation yields the 't Hooft (or, if one wishes, "traditional") large N_c limit (TLNC limit), while using the anti-symmetric representation yields the new orientifold (or "other") large N_c limit (OLNC limit).

The OLNC limit has a number of attractive features from a theoretical perspective. It is inspired by and related to supersymmetric Yang-Mills theory, and for one flavor allows one to apply some of the powerful analytic tools and results of supersymmetric Yang-Mills theory to QCD. However, it is important to note that the OLNC has important differences from the TLNC. While non-planar diagrams are suppressed in the OLNC limit (similarly to the TLNC limit), quark loops are not suppressed in the OLNC limit, since they, like gluons, carry two color indices. This alters the nature of the large N_c scaling in the theory. Most significantly it implies that an n-meson vertex scales with N_c differently in the two expansions:

$$\begin{aligned}\Gamma_n &\sim N_c^{2-n} \quad \text{(OLNC)}\\ \Gamma_n &\sim N_c^{1-n/2} \quad \text{(TLNC)}\,.\end{aligned} \tag{1.1}$$

These scaling relations show that in the OLNC limit, mesons behave analogously to glueballs in the TLNC limit[12]. This is as one would expect, since in the OLNC limit both quarks and gluons carry two color indices.

Apart from the above difference in the scaling of meson interactions, there is another important distinction between the OLNC and the TLNC limit. Since quark loops are not suppressed in the OLNC limit, unlike the TLNC limit it does not impose the OZI rule. This has the disadvantage of not explaining a generic feature of hadronic phenomenology (that the TLNC limit explains quite neatly). However, it has the compensating virtue of not requiring large $1/N_c$ corrections in those situations where quark loops are important, such as in the $\eta' - \pi$ mass difference.

Witten [2, 4] showed that it is natural to make an identification between baryons and solitons, such as the Skyrmion, in the TLNC limit. The evidence for this was based on explicit calculations of the scaling of the baryon and meson masses and scattering amplitudes with N_c. It was seen that *all* properties of baryons and mesons scale with $N_c^{-1/2}$ in the same way as an analogous meson-based soliton model scales with a generic meson-meson coupling constant g. It is important to determine whether this baryon-soliton identification can be made in the new OLNC limit.

At first sight it appears that the identification does not work: the mass of baryons is usually thought to scale as N_c^1, while as pointed out by Armoni and Shifman[5], the mass of Skyrmions in the OLNC limit scales as N_c^2, creating an apparent contradiction. It is not hard to see that the Skyrmion mass scales as N_c^2. For illustration consider the simplest Skyrmion for two massless flavors. The Lagrangian density is given by

$$\mathcal{L}_S = \frac{f_\pi^2}{4}\mathrm{Tr}(L_\mu L^\mu) + \frac{\epsilon^2}{4}\mathrm{Tr}([L_\mu, L_\nu]^2)\,, \tag{1.2}$$

where the left chiral current L_μ is given by $L_\mu \equiv U^\dagger \partial_\mu U$, with $U \in SU(2)_f$ [11, 13]. The U field can be written as $U = \exp(i\vec{\tau}\cdot\vec{\pi}/f_\pi)$, where $\vec{\pi}$ is the pion field. Upon expanding the pion field in the Lagrangian one sees that the n-meson vertices agree with the generic

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scaling rules of Eq. (1.1) only if

$$\begin{aligned} &\epsilon \sim N_c^{1/2} \quad f_\pi \sim N_c^{1/2} \quad \text{(TLNC)} \\ &\epsilon \sim N_c^{1} \qquad f_\pi \sim N_c^{1}\, . \quad \text{(OLNC)} \end{aligned} \tag{1.3}$$

The mass of the Skyrmion depends only on the parameters f_π and ϵ; the standard variational treatment [13] yields a soliton mass given by $M_s = \overline{m}\epsilon f_\pi$ where $\overline{m}$ is a dimensionless number obtained from the solution of the variational equation. From the scaling behavior of f_π and ϵ in Eq. (1.3), one sees that the soliton mass scales as $M_s \sim N_c^2$. Moreover, it is quite easy to see that the scaling of the soliton mass with N_c^2 is generic; it does not depend on the details of the particular Skyrmion Lagrangian used.

However, Bolognesi [10] has shown that the discrepancy between a soliton mass scaling as N_c^2 and a baryon mass scaling as N_c^1 is due to a naive (and incorrect) expectation about the scaling of the baryon mass. In fact, Bolognesi showed that a color singlet baryon state in the OLNC limit must contain at least $N_c(N_c-1)/2 \sim N_c^2$ quarks [10]. This suggests that baryon masses should scale as N_c^2, not N_c^1, which eliminates the apparent inconsistency.

Bolognesi's observation that order N_c^2 quarks are required to make a baryon is clearly of paramount importance in the identification of baryons as Skyrmions in the OLNC limit. Moreover, ref. [10] notes that the coefficient of the Wess-Zumino-Witten term must be $N_c(N_c-1)/2$, as one would expect in order for the identification to be consistent. However, by itself this is not sufficient. Recall that Witten's identification of baryons as solitons in the TLNC limit required far more than the simple observation that a baryon had at least N_c^1 quarks. Rather it was based on the observations that

1. The total contribution to the mass of the baryon—including the energy of interaction between the quarks via (multiple) gluon exchange—is of order N_c^1;

2. The characteristic N_c scaling of all other observables of baryons and mesons (such as scattering amplitudes or form factors) is analogous to the scaling of the same quantities in soliton models, provided one scales g, the characteristic coupling in the soliton model, as $N_c^{-1/2}$.

In fact, these conditions were not demonstrated rigorously in ref. [2]. Rather, it was shown that 1) various typical classes of gluon exchange diagrams contributing to the mass scaled as N_c (counting the combinatoric factors) and 2) characteristic classes of diagrams associated with the various observables scaled appropriately once combinatoric factors were included.

The question addressed in this paper is whether hadronic properties in the OLNC limit have the same N_c scaling as the properties of solitons, with the characteristic coupling constant g in the soliton model scaling as $g \sim N_c^{-1}$. Such a scaling rule is consistent both with the baryon scaling as N_c^2, and with the meson-meson scattering amplitudes given in Eq. (1.1). It is not clear how to demonstrate this in a completely rigorous manner. However, a demonstration with a degree of rigor comparable to Witten's original analysis for the TLNC limit will presumably suffice to make a compelling case. The goal of this paper is to provide such a demonstration via the consideration of classes of diagrams in

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a manner analogous to ref. [2]. This would essentially complete the program begun in ref. [10] of establishing a Skyrmionic description of baryons in the OLNC limit.

If one follows the arguments in this paper, it will be obvious that all of the qualitative conclusions for scaling rules with N_c apply equally to the case in which the quarks are taken to be in the two-index symmetric representation. However, we focus on the anti-symmetric case since it corresponds to the physical world at $N_c = 3$; the symmetric case does not.

The generalization of Witten's analysis to baryons in the OLNC limit is not completely trivial; there is an important subtlety for baryons in the OLNC limit which is not present in the TLNC limit. The nature of the issue can be seen by looking at the one-gluon exchange contribution to the baryon energy. For the TLNC limit, Witten showed these contributions scale as N_c^1 (ref. [2]). In a representative diagram of two quarks interacting via a single gluon exchange, there are two gluon vertices which together contribute a factor of $1/N_c$, and a combinatoric factor of N_c^2 since each end of the gluon can connect to one of the N_c distinct quarks in the baryon.

A naive generalization of this reasoning to the OLNC limit suggests that there the one-gluon exchange contribution to the mass scales like N_c^3. There is again a $1/N_c$ factor for the gluon vertices, but in the OLNC limit case there are $N_c(N_c - 1)/2 \sim N_c^2$ species of quark and thus the combinatoric factor appears to scale as N_c^4. If the contribution of the one-gluon exchange contribution to the nucleon mass really does scale as N_c^3, it suggests that the baryon mass grows with N_c faster than N_c^2, apparently preventing an identification of baryons with the Skyrmions in the OLNC limit.

In this paper, we demonstrate that despite the apparent discrepancy above, the one-gluon exchange contribution to the baryon mass scales only as N_c^2. As will be seen, there is an important difference in the nature of one-gluon exchange in the two limits which ultimately resolves the apparent paradox involving the one-gluon exchange contribution to the baryon mass discussed above. Moreover, we show more generally that the contribution to the mass from *all* types of multiple gluon exchange diagrams scales as N_c^2. This is what is required to have the baryon mass scale as N_c^2, and thus to obtain precisely the behavior needed for the baryon to scale as a Skyrmion in the OLNC limit.

Similarly, we study characteristic diagrams contributing to numerous quantities associated with hadronic interaction and from these diagrams abstract the N_c scaling behavior. In particular, we consider the strength of the meson-baryon coupling (N_c^1), the baryon-meson scattering amplitude (N_c^0), baryon-meson scattering to a two-meson final state (N_c^{-1}), and the baryon-baryon coupling (N_c^2). These are precisely the scaling rules one would expect if the baryon were a Skyrmion.

Given these scaling results, we argue that one can view baryons as Skyrmions in the OLNC limit as well as in the TLNC limit. The fundamental difference between the two cases is that any quantity which scales as N_c^k in the TLNC limit scales as N_c^{2k} in the OLNC limit.

In the analysis that follows we will sometimes draw representative Feynman diagrams. Occasionally, where it is important to illustrate the color flow, we follow 't Hooft and use color-flow diagrams in which we draw gluons as two oppositely directed color lines. In the TLNC limit, quarks are represented by single fermion lines, while in the OLNC limit

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quarks are represented by doubled fermion lines pointing in the same direction, in order to reflect the fact that quarks now carry two color indices. The double-line representation for quarks in the OLNC limit will be used in both Feynman diagrams and color-flow diagrams.

The central focus of this paper is on baryons. However, the identification of baryons as solitons in a mesonic theory requires an understanding of the scaling rules in the meson sector encapsulated in Eq. (1.1). Moreover, the elucidation of some aspects of the mesonic sector is essential for clarifying the meson-baryon interaction. Accordingly, the next section will sketch the derivation of the scaling rules for the meson sector. Since these results are well known there is no need to be complete; we only attempt to provide enough detail to elucidate the main points. Next, we devote a short section to the discussion of a vital difference in the color-flow in one-gluon exchanges between two quarks in the TLNC and OLNC limits. This distinction will help resolve the apparent paradox involving baryon mass scaling that was discussed above. Following that section, we turn to the main focus of the paper: the scaling properties of baryons. We consider classes of diagrams which enable us to deduce the scaling of the baryon mass and various aspects of interactions of baryons with other hadrons. Finally, there is a brief concluding section.

2. Mesons

In this section we briefly review the large N_c scaling of meson interaction amplitudes in the TLNC and OLNC limits. While the results are well known, they are useful in what follows. Throughout the section, we first review how the analysis works for a given quantity in the TLNC limit, and then discuss the analogous derivation in the OLNC limit. To streamline the discussion, we examine simple quark loops as representatives of the leading order class of diagrams for each quantity we examine. This can be done without loss of generality as the inclusion of more complicated planar graphs clearly does not alter the result.

In both of the TLNC and OLNC limits, meson masses have the same scaling as quark masses, i.e., they scale as N_c^0. Our first step is to determine the N_c scaling of the matrix element for a current to create a meson.

We begin with the TLNC limit. Consider a quark loop with two currents carrying meson quantum numbers at the edges as a representative diagram for the two-point correlation function (figure 1(a) — the solid dots represent the currents). There are N_c^1 choices of color for the quark loop, so the diagram must scale as N_c^1 as a whole. Matching the N_c scaling of the diagrams with the meson picture, one sees that the amplitude for the current to create a meson must scale as $N_c^{1/2}$.

The analysis proceeds in an analogous manner for the OLNC limit; the only significant difference is that there are N_c^2 choices for the color loop in figure 1(b), and as a result each meson creation matrix amplitude scales as N_c^1 rather than $N_c^{1/2}$. At this point, we should note that up to constants of proportionality, f_π is the amplitude for the axial current operator to create a pion from the vacuum. The preceding analysis shows that $f_\pi \sim N_c^{1/2}$ for the TLNC limit while $f_\pi \sim N_c^1$ for the OLNC limit. This is precisely what is needed for consistency with the Skyrme Lagrangian as seen in Eq. (1.3).

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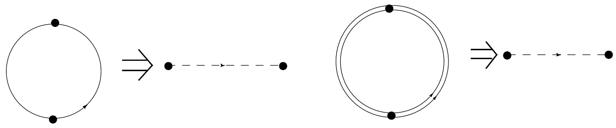
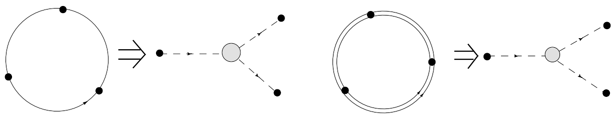


(a) TLNC limit (b) OLNC limit

Figure 1: Quark loops with two current insertions (as representatives of the class of leading order diagrams for the two-point function) and their associated hadronic content in terms of meson propagation.



(a) TLNC limit (b) OLNC limit

Figure 2: Meson decay diagrams. The relationships between quark loops (as typical members of the class of leading order diagrams) with three current insertions and the hadronic-level effective diagrams are illustrated.

Now consider the N_c scaling of the amplitude for the three-meson vertex which fixes the strength of a meson decaying into two mesons. In the TLNC limit (figure 2(a)), we again start with a quark loop, but this time with three current insertions, as a representative of the class of leading order diagrams for the three-point correlation function. At the hadronic level this diagram represents the creation of three mesons from the currents, with the mesons interacting via a trilinear meson-meson-meson vertex. The diagram as a whole still scales as N_c^1, but we know that each of the matrix elements scale as $N_c^{1/2}$. This means that the trilinear meson-meson-meson vertex must scale as $N_c^{-1/2}$. From this we conclude that the amplitude for meson decays scales as $N_c^{-1/2}$, while the width scales as N_c^{-1}, and thus mesons are stable at large N_c in the TLNC limit.

In the OLNC limit, the main difference is again the N_c^2 choices of color labels for the quark loop (figure 2(b)). It is not hard to see that this implies that the three meson vertex must scale as N_c^{-1} and its width therefore scales as N_c^{-2}; mesons are also stable in the OLNC limit. Note that the scaling relation for the three-meson vertex is consistent with Eq. (1.1).

It should be immediately clear from the preceding example how to generalize to the case of an interaction vertex for any number of mesons. Adding one more meson reduces the scaling by a factor of $N_c^{-1/2}$ for the TLNC limit and by a factor of N_c^{-1} for the OLNC limit. Taken together these immediately yield Eq. (1.1).

Note that the generic replacement rule for scaling that was given in the introduction, $N_c^k(\text{TLNC}) \to N_c^{2k}(\text{OLNC})$, holds throughout the meson sector.

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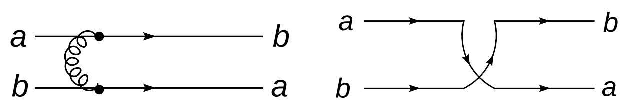
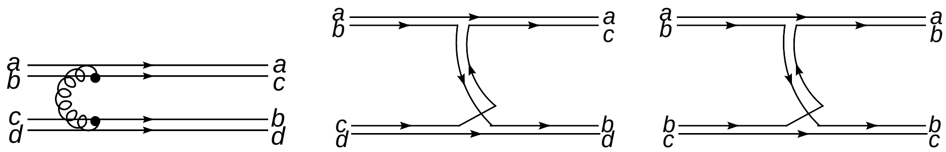


(a) Quarks in the fundamental representation

(b) Color flow for quarks in fundamental representation

Figure 3: One-gluon exchange between quarks in the fundamental representation. The colors a and b are switched by the exchange



(a) Quarks in anti-symmetric representation

(b) Generic color flow for quarks in anti-symmetric representation

(c) Color flow for gluon exchange with no change in color labels

Figure 4: One-gluon exchange between quarks in the anti-symmetric representation. The colors for the initial quarks ab and cd are generally, but not necessarily, distinct from the colors for the final quarks, ac and bd.

3. One-gluon exchange

As noted in the introduction, in order for there to be a possibility of identifying baryons with solitons in the OLNC limit, there must be a subtle distinction between the behavior in the TLNC limit and OLNC limit. The naive analysis of the one-gluon exchange contribution to the baryon mass gives a result consistent with the Skyrmion for the TLNC limit and a result apparently inconsistent for the OLNC limit. The origin of this discrepancy can be traced to the nature of gluon exchange between quarks in the two cases. In this section we focus on elucidating the differences in one-gluon exchange between two quarks in the TLNC and OLNC limits.

Consider one-gluon exchange between two quarks in the TLNC limit (figure 3(a)), where the quarks are taken to be in the fundamental representation and thus are labeled by a single color. The key point is that the effect of the gluon exchange on the quark content is simply to switch around the color labels of the two quarks (a and b in the figure), *i.e.*, after the exchange one has quarks with the same colors as before the exchange. The reason for this is clear from the color flow diagram of figure 3(b).

In contrast, consider a one-gluon exchange for two quarks in the anti-symmetric representation relevant for the OLNC limit (figure 4(a)), where each quark is labeled by *two* color indices. Note that while the total color of the state is preserved by the interaction (one has fundamental colors a, b, c and d both in the initial and final state), the color labels of the individual quarks are generally altered. In the case illustrated in figure 4(a), initially

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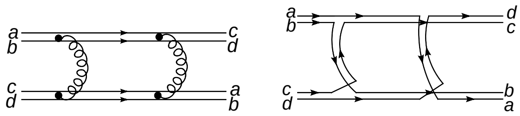

(a) Two-gluon exchange

(b) Color flow for two-gluon exchange

Figure 5: Two-gluon exchange graphs for quarks in the anti-symmetric representation. The quarks can have generic initial color labels and suffer no change in final quark color labels

one has quarks of the ab and cd varieties, but after the interaction there is one quark with ac and one with bd. The reason for this is clear from the color flow diagram in figure 4(b). Thus, unlike the situation with quarks taken to be in the fundamental representation, as in the TLNC limit, gluon exchanges typically alter the color labels of quarks taken to be in the anti-symmetric representation, as in the OLNC limit.

This fact plays a critical role in combinatoric counting at large N_c. If we restrict ourselves to situations in which the colors of the initial and final quarks must be the same (up to a permutation), then a one-gluon exchange in the OLNC limit requires a constraint on the type of quarks which participate. In particular, they have to share one of their two color indices. For example, in the diagram in figure 4(a), the restriction is for $a = d$ (recalling that the labels ba and ab are equivalent for an anti-symmetric representation). This restriction will play a key role in reducing combinatoric factors when considering the scaling of baryonic quantities.

Two quarks can also exchange a gluon (e.g., a $b\bar{b}$, as in figure 4(c)) and undergo no changes or permutations in color labels. In such situations, the gluons must be in the Cartan subalgebra — that is, the diagonal subalgebra — of the algebra of $SU(N_c)$ [16]. It is non-trivial to show such gluons in an 't Hooft style double-line diagram, but the imposition of the tracelessness condition for the $SU(N_c)$ algebra is a $1/N_c$ suppressed effect, and thus one can simply work with gluons in the algebra of $U(N_c)$ to leading order in $1/N_c$[17] — which is what we do in figure 4(c).

Of course it is possible for two quarks with no shared color labels to interact with no change of color labels, but this generally requires a two-gluon exchange (figure 5(a)). It is clear that in a certain sense, the case of two-gluon exchanges between two quarks in the OLNC limit is analogous to one-gluon exchange in the TLNC limit. Again, the reason this works is easily seen in the color-flow diagram of figure 5(b). This fact also plays an important role in the scaling at large N_c, since the two-gluon exchange diagrams have an extra factor of $g^2 \sim N_c^{-1}$ compared to one-gluon exchange.

The preceding illustrates the central distinction between the nature of gluon exchange between quarks in the fundamental and anti-symmetric representations. It makes clear that we cannot simply copy Witten's combinatoric analysis developed for the TLNC limit for the OLNC limit analysis. Instead, we must modify it suitably to account for the differences in one-gluon exchanges in the two limits. Once this is taken into account, it is straightforward

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to show that the baryon quantities in the OLNC limit do in fact scale with N_c in a manner consistent with a Skyrmion.

4. Baryons

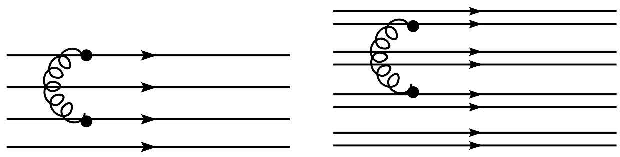
4.1 Baryon mass

In the traditional 't Hooft large N_c, limit baryons are antisymmetric, color-singlet combinations of N_c quarks (plus associated gluons and those quark-antiquark contributions which arise through "z-graphs" without closed quark loops [15]). The quarks have a fixed mass of order N_c^0, yielding a contribution to the baryon mass that scales as N_c^1; similarly, the kinetic energy of the quarks is a one-body operator and its contribution to the baryon mass also scales as N_c^1. Thus, it is is natural to assume that the baryon mass scales as N_c^1. For consistency, the contributions to the baryon mass from gluon exchange must also scale like N_c^1. It is not very difficult to verify that this is indeed the case.

Witten showed that in order to investigate gluon-exchange contributions to the baryon mass, the relevant quantities to study are the quark-line connected diagrams (the disconnected ones arise through exponentiation of the Hamiltonian) [2]. Consider, as a simple example, the one-gluon interaction between a pair of quarks in the baryon as illustrated in figure 6(a). As discussed briefly above, this contribution scales as N_c^1. Recall that any two quarks in a baryon can interact in this way, since they simply exchange color indices in the interaction, keeping the baryon a color singlet. The two quark-gluon vertices together scale as $(N_c^{-1/2})^2 = N_c^{-1}$. There are N_c^1 choices for the first quark involved and another N_c^1 choices for the second one, giving a total combinatoric factor of N_c^2. It follows that such diagrams are of order N_c^1.

Quark-line connected diagrams involving more than two quarks do not change this conclusion because connecting an additional quark to the diagram requires adding two new gluon vertices, for a factor of N_c^{-1}, and a combinatoric factor of N_c^1 from the sum over colors. As a result additional connected quarks only add factors of N_c^0 to such self-interaction diagrams. This reasoning can easily be cast into the form of an argument by induction, and a generalization of this idea will be used in the discussion of the OLNC limit below.

Inserting additional gluons which connect to pre-existing gluons does not alter the counting. By standard arguments an additional gluon will at most add a closed color loop in the sense of 't Hooft diagrams thereby adding a power of N_c; this is compensated for by two coupling constants at N_c^{-1} yielding no change in the N_c counting (this is the analog of the planar diagrams from the meson case). Depending on the topology of the diagram, additional gluons may not add a color loop, in which case their graphs are suppressed in the $1/N_c$ expansion (these are the non-planar graphs). Additional quark loops do not add a color loop but cost a power of $1/N_c$ from the vertices and are thus always suppressed. From these considerations, we see that in the TLNC limit the general gluon-exchange contribution to the baryon mass really is of order N_c^1. This is consistent with the baryon mass scaling as N_c^1.



(a) TLNC limit (b) OLNC limit

Figure 6: Typical one-gluon exchange diagrams that contribute to the baryon mass. The letters correspond to quark color labels before and after the gluon exchange.

In the OLNC limit the situation is somewhat more complicated. As shown by Bolognesi [10], in this limit baryons are an antisymmetric combination of $N_c(N_c-1)/2 \sim N_c^2$ quarks, each of which now carries two color indices. Since each quark still has a mass and a kinetic energy of order N_c^0, this means that in this limit the baryon mass should scale as N_c^2. However, for this to be true, the contribution to the mass from gluon-exchange interactions between the quarks must also scale as N_c^2 in the OLNC limit.

First, consider a representative diagram of a one-gluon interaction between two two-index quarks in an OLNC limit baryon (figure 6(b)). As noted in the introduction, a naive recapitulation of the reasoning used in the TLNC case leads to the conclusion that such diagrams scale as N_c^3: there are two gluon vertices, which together scale as N_c^{-1}, and N_c^2 choices for each of the two participating quarks, yielding a complete diagram that scales as $N_c^{-1}N_c^4 = N_c^3$. This is clearly inconsistent with the baryon mass scaling like N_c^2.

In fact, a more careful analysis shows that one-gluon interaction diagrams in the OLNC limit scale as N_c^2. The basic reason was foreshadowed in the preceding section: as in the TLNC limit, the interacting quarks swap a color index through the interaction, but because each quark now carries two color indices, there are restrictions on which quarks can interact in this way within a baryon. For example, suppose the interacting quarks are labeled with color indices ab and cd. After exchanging a $b\bar{c}$ gluon, they become labeled with the indices ac and bd (figure 7(a)). However, since the baryon is an antisymmetric combination of all possible two-color labeled quarks, after such an interaction the baryon would 'lose' the ac and bd quarks by antisymmetry, as well as the ab, cd quarks. Such a final state must vanish. This forces us to conclude that quarks which do not share at least one color label cannot interact directly via a one-gluon exchange in a baryon.

However, if two quarks *do* share a color label, then a direct interaction between them will survive. For example, two quarks labeled ab and bc can interact via the exchange of an $a\bar{c}$ gluon, and will have color labels cb, ba after the interaction (figure 7(b)) — the color labels are simply permuted. As desired, after the interaction the baryon still consists of an antisymmetric combination of all possible two-color labeled quarks.

A one-gluon exchange within a baryon (see, e.g., figure 4(c)) that does not alter or permute any color labels and is allowed by the antisymmetry condition is also possible[16]. As discussed in the preceding section, this involves gluons in the Cartan subalgebra of $SU(N_c)$. In such diagrams, the involved quarks must share some color labels, so their N_c

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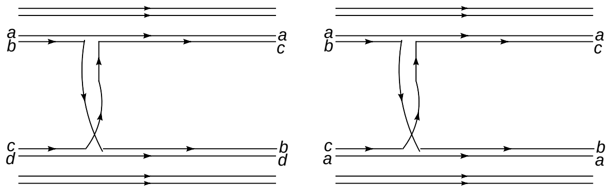
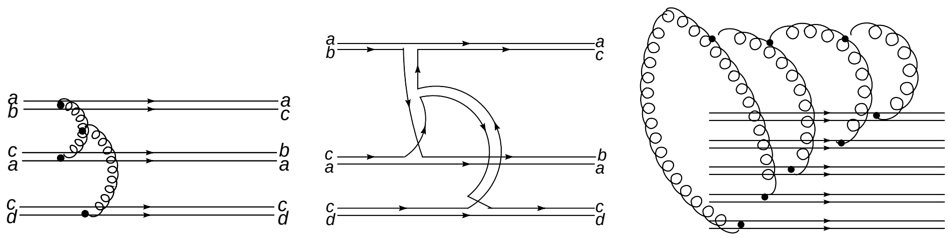


(a) Forbidden interaction

(b) Allowed interaction

Figure 7: Not every interaction between two quarks within a baryon is allowed in the OLNC limit.



(a) Feynman diagram for an interaction between two unlike quarks *ab* and *cd*

(b) Color flow for an interaction between two unlike quarks *ab* and *cd*

(c) A typical many-quark interaction

Figure 8: Gluon-exchange interactions in a baryon between multiple quarks in the OLNC limit.

scaling is the same as those of the other diagrams involving one-gluon exchange.

From these considerations we see that only quarks that share a color label can interact via a one-gluon exchange in a baryon in the OLNC limit. Consider now the N_c scaling of a diagram of such an interaction (figure 6(b)) in the OLNC limit. There are two quark-gluon vertices, for a factor of $(N_c^{-1/2})^2 = N_c^{-1}$. There are N_c^2 choices for the first quark involved, but only N_c^1 choices for the second because it must share a color label with the first quark, giving a combinatoric factor of N_c^3. Thus the entire diagram scales as N_c^2.

We note that two quarks in a baryon that share no color indices *can* interact with each other, but the interaction must involve more than one gluon exchange. If two quarks exchange two gluons directly (as in figure 5(a)), the four gluon vertices will give a factor of N_c^{-2}, and the N_c^2 choices for the labels of each of the two quarks will result in a N_c^2 scaling for the interaction. Alternatively, two quarks with unlike labels may interact via gluon exchanges with an intermediary third quark, which must share some indices with both of the unlike quarks, as in figure 8(a).

To show that gluon exchanges contribute at most N_c^2 to the baryon mass, we must demonstrate that diagrams with an arbitrary number of interacting quarks within a baryon scale as N_c^2. To show this, we will construct an argument by induction that shows that diagrams with q interacting quarks (figure 8(c)) scale as N_c^2 at large N_c. The argument by induction is essentially based on the idea that one can build a diagram with $(q+1)$

interacting quarks by adding a quark to some q-quark diagram, and the observation that such an addition does not change the N_c scaling of the diagram.

As the base case (that is, $q = 2$), we have already shown above that diagrams with two interacting quarks scale as N_c^2. Next, observe that any leading-order diagram with $q+1$ interacting quarks can be constructed from some q-quark diagram by connecting (via one or more gluons) an additional quark. As the inductive step, suppose that the q-quark diagram scales as N_c^2. We can connect a new $(q+1)^{st}$ quark to the diagram in one of three ways: either by a one-gluon connection to a quark in the q-quark diagram, by a one-gluon connection to a gluon in the q-quark diagram, or by a two-gluon connection to a quark in the q-quark diagram.

The first two cases above are identical as far as the topology of color flow is concerned, as an inspection of figure 8(b) makes clear. Therefore, we can consider only the cases of direct quark-quark connections, without loss of generality. Since the new quark connects via gluon exchange to a quark in the q-quark diagram, the situation is reduced to that of the base case of two interacting quarks.

If only one gluon is exchanged (with a factor of N_c^{-1} from the two new coupling constants), the new quark must share a color index with the quark with which it is interacting, yielding a combinatoric factor of N_c^1. Alternatively, if two gluons are exchanged (with a factor of N_c^{-2} from the four new coupling constants), the new quark need not share any color indices with the quark with which it is interacting, yielding a combinatoric factor of N_c^2. In either case, the scaling of the $(q+1)$-quark diagram is the product of the scaling of the q-quark diagram, N_c^2, and a factor of either $N_c^{-1} N_c^1 \sim N_c^0$, or $N_c^{-2} N_c^2 \sim N_c^0$. Thus we see that a general $(q+1)$-quark diagram scales as N_c^2 in the OLNC limit. This completes the argument by induction, and we conclude that any diagram with q interacting quarks scales as N_c^2 at leading order.

Of course, diagrams beyond the class considered above can contribute. For example, additional gluons can connect between the gluons in flight yielding closed gluon loops. However, such additional gluon loops will not alter the N_c counting. As in the case of the TLNC limit, adding a gluon to a diagram can at most add a closed color loop in the sense of an 't Hooft diagram, adding a factor of N_c which is compensated by a N_c^{-1} factor from the additional vertices. This yields either an unchanged N_c scaling or a suppression.

Unlike the TLNC limit, closed quark loops are not suppressed in the OLNC limit. Due to their two-index nature they behave analogously to gluons. Depending on the topology of the diagram, quark loops can add at most one new color loop, which is exactly compensated for by the N_c^{-1} factor due to the new vertices. Thus, while quark loops are not suppressed, they also do not alter the leading N_c counting.

As a result of these considerations, it is apparent that the total energy of interactions between quarks due to the exchange of gluons is of order N_c^2. Thus we see that the baryon mass consistently scales as N_c^2. This is consistent with the known scaling of the soliton mass, which is also N_c^2 in the OLNC limit.

4.2 Scattering

Our goal in this subsection is to show that the scaling rules for scattering amplitudes and

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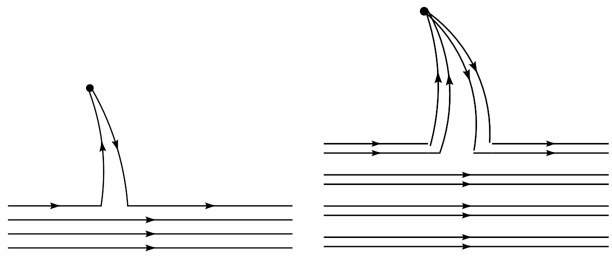


(a) TLNC limit (b) OLNC limit

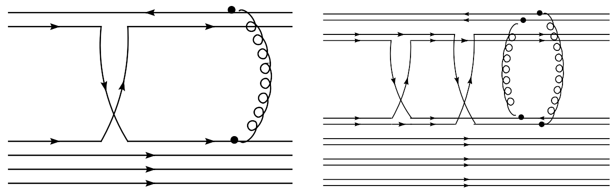
Figure 9: Representative diagrams for the meson-baryon coupling.

coupling constants between baryons and mesons in the OLNC limit work analogously to the parallel quantities in the TLNC limit with the standard substitution $N_c^k \to N_c^{2k}$ required for the consistency of the Skyrmion picture.

We begin with an examination of the baryon-meson vertex. First, consider typical diagrams representing a baryon emitting a meson (figures 9(a), 9(b)) in the TLNC and OLNC limits. The dot represents a current with the quantum numbers of some meson. One can add to these "skeletons" various gluon insertions (and quark loops for the case of the OLNC limit) without altering the basic N_c counting rules. The amplitudes for coupling to a meson, as opposed to the current itself, will be controlled by the amplitude for the creation of an extra meson, which as shown in Sec. 2 scales like $N_c^{-1/2}$ and N_c^{-1} in the TLNC and OLNC limits respectively. Thus, one generically expects that the meson-baryon coupling constant will scale as $N_c^{1/2}$ (TLNC limit) or N_c^1 (OLNC limit). This is consistent with the identification of a baryon as a soliton: the soliton-meson coupling generically scales as $1/g$. Thus, as is expected, the scaling matches provided $g \sim N_c^{-1/2}$ (TLNC limit) or $g \sim N_c^{-1}$ (OLNC limit).

Next consider meson-baryon scattering. First consider the TLNC limit. A characteristic diagram contributing to the process (figure 10(a)) is the exchange of a quark between the baryon and the meson; following this exchange there must be a gluon exchange to keep the baryon and meson separately color singlets. In such a graph, there are two gluon vertices (for a suppression of $1/N_c$), and a combinatoric factor of N_c since N_c different quarks in the baryon can participate in the exchange. As a result, typical diagrams for baryon-meson scattering scale as $(N_c^{-1/2})^2 N_c^1 = N_c^0$. This result is consistent with meson-soliton scattering with the standard identification since the meson-scattering amplitude is independent of g at large g.

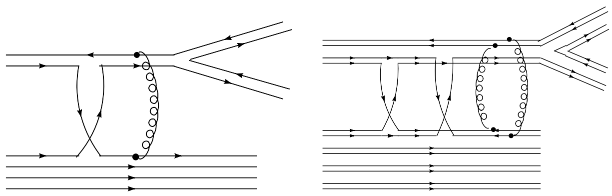
Now consider an analogous diagram in the OLNC limit (figure 10(b)). As before, the interaction takes the form of a quark exchange. However, since the quarks now carry two color indices, there must be at least two gluons exchanged in order to keep the baryon and meson separately color singlets . As a result, there are four gluon vertices in a representative diagram, contributing a total of $(N_c^{-1/2})^4 = N_c^{-2}$, and a combinatorial factor of N_c^2 due to the sum over the possible color labels for the quark in the baryon participating in the interaction. The complete diagram thus scales as N_c^0, just as before. Again this is consistent



(a) TLNC limit, Feynman diagram

(b) OLNC limit, Feynman diagram

Figure 10: Representative diagrams contributing to meson-baryon scattering.



(a) TLNC limit, Feynman diagram

(b) OLNC limit, Feynman diagram

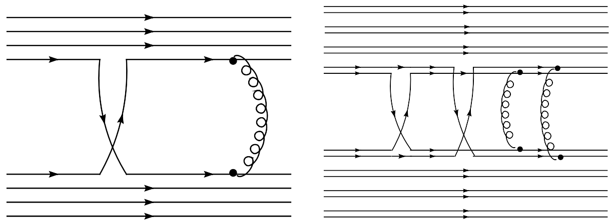
Figure 11: Representative diagrams for meson-baryon scattering with a two-meson final state.

with a soliton description.

Next consider a scattering process in which an incident meson on a baryon yields a final state with two mesons. We first review the situation in the TLNC limit (figure 11(a)). An incoming meson interacts with a baryon as in the meson-baryon case (by a quark exchange plus a gluon interaction), and then decays into two outgoing mesons. The first part of this interaction, involving the baryon, scales as N_c^0 as argued above. For the second part, involving a meson decay, one may recall from above (in Sec. 2) that the amplitude for such a process scales like $N_c^{-1/2}$. Thus the complete diagram scales as $N_c^{-1/2}$ in the TLNC limit. As before, this is consistent with a soliton description, in which such a process scales with the generic meson coupling constant, g, as $g^1 \sim N_c^{-1/2}$ in the TLNC limit.

For an analogous diagram for the OLNC limit, we claim that diagrams that show scattering with an initial meson on a baryon yielding a two-meson final state scale as N_c^{-1} (figure 11(b)). As before, the baryon-meson interaction scales as $(N_c^{-1/2})^4 N_c^2 = N_c^0$. Recalling the result for meson decays in the OLNC limit, we see that the meson decay part of the diagram now scales as N_c^{-1}. Thus the full diagram scales as N_c^{-1}. Again, this is consistent with a soliton description, since in the OLNC limit the generic meson coupling constant g scales as N_c^{-1}.

Finally we consider baryon-baryon scattering. As noted by Witten, the kinematics of this situation are peculiar. Since the mass grows with N_c, the description of baryon-baryon scattering at large N_c ultimately turns out to be smooth in the limit where the mass and momentum go to infinity at large N_c, in such a manner that the velocity p/M



(a) TLNC limit (b) OLNC limit

Figure 12: Representative baryon-baryon interaction diagrams.

remains fixed[2]. It should be noted that it is precisely in this limit that generic soliton models have well-defined scattering amplitudes as $g \to 0$. Secondly, the natural way to describe the situation is through the overall strength of interactions during the process — essentially the non-relativistic potential between the baryons [18, 19] which will ultimately be seen to be strong — and not through the scattering amplitude. As noted by Witten, if the energy of interaction is comparable to the incident kinetic energy, the two can play off each other in a smooth way. Thus, the quark-line connected diagrams between baryons should be interpreted in terms of the potential. The iteration of these between propagating individual baryons gives the full amplitude.

In the TLNC limit, a representative diagram for this is figure 12(a). The two baryons exchange a quark and also a gluon in order to stay as individual color singlets. There are N_c choices for each of the two quarks to participate in the interaction, and two gluon vertices, meaning the entire diagram scales as $N_c^2(N_c^{-1/2})^2 = N_c^1$. This means that the baryon-baryon potential is of the same scale as the baryon mass and kinetic energy (with fixed velocity).

The situation in the OLNC limit (figure 12(b)) is analogous, but because each quark now carries two color labels, two gluons must be exchanged to keep the baryons color singlets. The N_c scaling of such scattering diagrams is simply $(N_c^2)^2(N_c^{-1/2})^4 = N_c^2$. Again, we see that the baryon-baryon potential has the same scaling as the baryon kinetic energy. Again, this result is fully consistent with soliton-soliton scattering, where the energy of interaction between the solitons during the collision is of order $1/g^2$.

5. Conclusions

Using the results of [10] for the allowed representations for baryons in the OLNC limit, we have shown that the baryon mass scales as N_c^2; this holds even when quark-quark interactions through gluon exchange are taken into account. In doing so, we have resolved the apparent paradox that a naive generalization of Witten's counting for one-gluon exchange appears to scale as N_c^3. More generally, we have shown how to generalize Witten's analysis of baryon and meson behavior to the OLNC limit, and demonstrated that the replacement rule $N_c^k \to N_c^{2k}$ is justified for all of the representative diagrams.

From this analysis, one can conclude that all of the arguments for identifying baryons with solitons (such as the Skyrmion) in the 't Hooft large N_c limit apply to baryons in the orientifold large N_c limit. In general, to use the original Skyrme model to model baryons in the orientifold large N_c limit, one must simply scale $f_\pi^2 \sim N_c^2$ and $\epsilon^2 \sim N_c^2$ from Eq. (1.2), and also scale the WZW coefficient as $n \sim N_c^2$ in accordance with the replacement rule above. This will ensure that all of the generic hadronic scaling rules behave correctly.

We note, however, that *the* Skyrme model (i.e., Skyrme's original model) is *not* justified in large N_c limit. What is presumably justified is *a* Skyrme-type soliton model with an arbitrary number of fields and arbitrarily complex interactions. The justification for such a model based on generic scaling rules for QCD in the OLNC limit is essentially the same as in the TLNC limit.

We should also note that of course the Skyrmion encodes more than just the generic scaling rules, as it also encodes large N_c scaling rules associated with spin and flavor. Relations between observables sensitive to spin and flavor in Skyrme-type modes but independent of the dynamical details of the particular model were noted early on by Adkins and Nappi [14]. Subsequently, it was noted first by Gervais and Sakita[20] and then developed in considerable detail by Dashen and Manohar [21] and Dashen, Jenkins and Manohar [22] that such relations stem from large N_c consistency conditions. Since the key to this derivation is the fact that the pion-nucleon coupling constant grows with N_c while the pion-nucleon scattering amplitude does not, one expects that all of these relations will go through without essential change from the TLNC limit to the OLNC limit, again supporting the Skyrmion picture.

Although it is clear that a Skyrme-type model is capable of describing both limits (with the parameters having a different scaling with N_c as one goes from one to the other), there clearly are distinctions between baryons in the TLNC limit and the OLNC limit stemming from the non-suppression of quark loops in the OLNC limit. How these distinctions may be manifest in Skyrme-type models will be the subject of a future publication.

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